# Lecture 15

Next: [Finish the CW complex theorem.
Work out some examples more explicitly.

$G$ s.s. $\mathbb{C}$ Lie group. $G \supset B \supset H \rightsquigarrow \Phi \supset \Phi^+ \supset \Delta$ and $W = N(H)/H$

<u>Cell theorem</u>. $\forall\, w \in W$, $Bwx_0 \subset G/B$ is iso to $\mathbb{C}^k$ for some $k$.

Recall $B$ is assoc to $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Let $B^-$ corresp $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$.

Similarly there is a closed subgrp. $\mathcal{U} \subset B$ assoc to $\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathcal{U}^- \subset B^-$ assoc to $\bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$.

<u>Thm</u>. $B$ diffeo to $H \times \mathcal{U}$ by the prod map, $H \times \mathcal{U} \to B$

$H \cong (\mathbb{C}^*)^{\mathrm{rk}(G)}$

$\mathcal{U} \cong \mathbb{C}^N$ by the exp map. $N = \#\Phi^+$

Further, let $\mathcal{U}_\alpha \cong \mathbb{C} \subset B$ denote $\exp(\mathfrak{g}_\alpha)$. Then $\mathcal{U}_\alpha$ closed and $\forall$ orderings $\Phi^+ = \{\alpha_1, \dots, \alpha_N\}$, prod

$$\mathcal{U}_{\alpha_1} \times \cdots \times \mathcal{U}_{\alpha_N} \to \mathcal{U}$$ is biholo.

Same for $\mathcal{U}' \subset \mathcal{U}$ assoc to $+$-closed subset of $\Phi^+$.

i.e. $\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$ is uniquely $\begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & y \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ & 1 & z \\ & & 1 \end{pmatrix}$

(See Borel, LAG.) $\square$

<u>Lem</u>. $H$ fixes $wx_0$ for all $w \in W$.

<u>Pf</u>. $hwx_0 = hwB = wh'B = wB$ as $h' \in H \subset B$. $\square$

<u>Fact</u>. $\mathrm{Fix}(H) = \{wx_0\}$ so "any two Borels contain a torus" means there is a finite set $\mathrm{Fix}(H) = \pi \subset G/B$ s.t. $\forall\, x, y \in G/B$ $\exists$ $g$ s.t. $g\pi \supset \{x, y\}$. $\square$

So $\mathrm{Stab}_B(wx_0)$ contains $H$. Then ask if any elts of $\mathcal{U}$ lie there.

$\mathrm{Stab}_G(wx_0) = wBw^{-1}$. So if $b \in B \cap wBw^{-1}$ then $b \in \mathrm{Stab}_B(wx_0)$.

let $\Phi_w^+ = \{\alpha \in \Phi^+, \ w\alpha \in \Phi^-\}$, let $\Gamma_w^+ = \{\alpha \in \Phi^+, \ w\alpha \in \Phi^+\}$

then $\Phi = \Phi_w^+ \cup \Gamma_w^+$ disjointly.

and $\underset{\hat{B}}{\mathcal{U}} \overset{(\text{diff})}{\cong} \mathcal{U}_w V_w$ where $\mathcal{U}_w = \exp\left(\bigoplus_{\alpha \in \Gamma_w^+} \mathfrak{g}_\alpha\right)$ $V_w = \exp\left(\bigoplus_{\Phi_w^+} \cdots\right)$

and $B = H\mathcal{U}_w V_w$. (diff)

<u>Thm</u>. $\mathcal{U}_w = \mathcal{U} \cap w\mathcal{U}w^{-1}$ $\quad V_w = \mathcal{U}^- \cap w\mathcal{U}w$

$\mathrm{Stab}_B(wx_0)$ is the group gen by $H$ and $\mathcal{U}_w$, which is diffeo to the product (but not iso).

□

<u>Cor</u>. The orbit $Bwx_0$ is iso to $B/H\mathcal{U}_w \cong V_w \cong \mathbb{C}^{\# \Phi_w^+}$

<u>Rest of CW theorem</u>: Why does this $(\mathbb{D}^{2k})^\circ \to G/B$ extend ctsly to $\mathbb{D}^{2k} \to G/B$ that is the orbit closure?

<u>Bott-Samelson</u>. Let $\mathcal{U}_\alpha^- = \exp(\mathfrak{g}_{-\alpha})$

Then $\forall \alpha \in \Phi^+$, $\mathcal{U}_\alpha$ and $\mathcal{U}_\alpha^-$ generate a subgroup of $G$ loc iso to $SL_2\mathbb{C}$. Call it $S_\alpha$.

$\mathfrak{g}_{-\alpha} \oplus \mathfrak{l} \oplus \mathfrak{g}_\alpha$

Let $B_\alpha^- \subset S_\alpha$ be the Borel containing $\mathcal{U}_\alpha^-$.

Then $S_\alpha / B_\alpha^- \cong \mathbb{CP}^1$ with $\mathcal{U}_\alpha$ corresp to aff chart of it

They show $\prod_{\alpha \in \Phi_w^+} S_\alpha / B_\alpha^-$ maps naturally to $G/B$

with $\prod \mathcal{U}_\alpha$ corresp to $B \cdot wx_0$.

Why enough. There is a CW structure on $(\mathbb{CP}^1)^n$ with a single $(2n)$-cell and boundary $\bigcup_i (\mathbb{CP}^1)^i \times pt \times (\mathbb{CP}^1)^{n-1-i}$

Terminology $\ell(w) := |\Phi_w^+|$ length of $w$ (rel to $\Delta$)

Say $w \le w'$ if $\Phi_w^+ \subset \Phi_{w'}^+$

Explicitly for $SL_n\mathbb{C}$: $W = S_n$. $\Phi_\sigma^+ = \{e_i - e_j \mid \sigma(i) > \sigma(j)\}$ $\overset{i<j}{}$

$\ell(\sigma) = \#$ "inversions"

$C_w$ is called the Schubert cell assoc to $w$.
$\cong \mathbb{C}^{\ell(w)}$

$\bar{C}_w = X_w$ is called the Schubert variety (projective, irreducible var $/\mathbb{C}$)

Explicitly for $SL_n\mathbb{C}$?

$F = gB$ represents the flag span$(v_1)$, span$(v_1, v_2)$, ...
where $g = (v_1 | v_2 | \cdots)$

$x_0 = I_{n\times n} B$ represents the coordinate flag. $E$

The left action of $B$ corresp to upward row ops.

Preserves $\dim F_i \cap E_j$ $\forall i, j$

e.g. $F_i \cap E_2$ means comp of $v_1 \text{--} v_i$ of form $\begin{pmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Exercise. $B\sigma x_0 = \{$flags $F$ s.t. $\dim F_i \cap E_j = \dim F(\sigma)_i \cap E_j\}$.

So $C_\sigma = \{$flags $F$ s.t. $\dim F_i \cap E_j = \#\{k \mid 1 \le k \le j,\ \sigma(k) \le i\}\}$

$X_\sigma = \{$ ——— ——— $\ge$ $\}$

Expected dimension matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$d_{23} = 2$. $F_2 \cap E_3$ has dim $= 2$

(in 4-space, generic is 1)